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This consistency is established by the vanishing of the determinant

$$\Delta \equiv \begin{vmatrix} m(n-1) & q(r-1) & mq(n-r) \\ n(m-1) & r(q-1) & nr(m-q) \\ mn-1 & qr-1 & mn-qr \end{vmatrix};$$

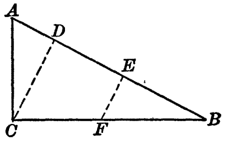
which proves the proposition.

Note.—Solutions of this problem appeared in the January and May numbers of the MONTHLY, presenting two different methods of attack. The one here given presents still a different method.

EDITORS.

459. Proposed by C. N. SCHMALL, New York City.

In a right triangle ABC, right-angled at C, a point F is taken in the side CB, and perpendiculars CD and FE are dropped on the hypothenuse AB.



Prove that

$$AD \cdot AE + CD \cdot EF = \overline{AC}^2.$$

SOLUTION BY HERBERT N. CARLETON, West Newbury, Mass.

Triangles ACD, CDB, EFB, and ACB are similar. Hence,

$$AD : CD = EF : EB;$$

whence

$$AD \cdot EB = CD \cdot EF.$$

Similarly

$$AD : AC = AC : AB;$$

whence

$$\overline{AC}^2 = AB \cdot AD = (AE + EB)AD = AD \cdot AE + AD \cdot EB.$$

Substituting for $AD \cdot EB$, we have

$$\overline{AC}^2 = AD \cdot AE + CD \cdot EF.$$

Also solved by A. M. HARDING, J. W. CLAWSON, ELMER SCHUYLER, FRANK IRWIN, WALTER C. EELLS, L. G. WELD, J. A. CAPARO, A. H. HOLMES, NATHAN ALTSHILLER, G. W. HARTWELL, HORACE OLSON, and HARMON ANNING.

460. Proposed by J. W. CLAWSON, Ursinus College, Pa.

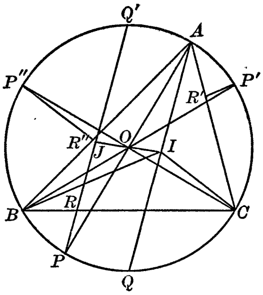
ABC is a triangle, O and I the centers of the circum- and in-circles respectively, and I', I'', I''' the centers of the three escribed circles. If AO, BO, CO meet the circumcircle in P, P', P'' respectively, and PR, $P'R'$, $P''R''$ are drawn parallel respectively to AI, BI, CI to meet BC, CA, AB respectively in R, R', R'', prove that: (1) PR, $P'R'$, $P''R''$ are concurrent, say at J. (2) $JO = OI$. (3) JI', JI'', JI''' are perpendicular respectively to BC, CA, AB. (4) AR, BR', CR'' are concurrent.

I. SOLUTION BY THE PROPOSER.

(1) Extend IO its own length to J. Join JP. Then $\triangle$'s AOI, POJ are congruent. Hence, $\angle APJ = \angle PAI$. Hence J lies on PR. Similarly, J can be shown to lie on $P'R'$ and on $P''R''$.

(2) By construction, $JO = OI$.

(3) Extend AI to meet the circumcircle at Q. Then Q is the center of a circle passing through B, C, I, I'. Hence, Q bisects II'. Also O bisects IJ. Hence, $I'J$ is parallel to OQ. But OQ is perpendicular to BC, since arc BQ = arc QC. Therefore, JI' is perpendicular to BC.



Similarly, JI'' and JI''' are perpendicular to CA and AB, respectively.

(4) Extend PJ to meet the circumcircle at Q'. It is easily shown that Q' is the opposite extremity of the diameter through Q. Then $\angle BPQ' = \angle Q'PC$.

$$\therefore \quad \frac{BR}{RC} = \sqrt{\frac{4r^2 - c^2}{4r^2 - b^2}}, \qquad \frac{CR'}{R'A} = \sqrt{\frac{4r^2 - a^2}{4r^2 - c^2}}, \qquad \frac{AR''}{R''B} = \sqrt{\frac{4r^2 - b^2}{4r^2 - a^2}},$$

where r = radius of circumcircle, and a, b, c are the sides of the triangle.

Multiplying,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = 1.$$

Hence, by the converse of Ceva's Theorem, AR, BR,' CR'' are concurrent.

II. Solution by N. P. Pandya, Sojitra, India.

(1) The quadrilateral $ABPP'$, having as diagonals diameters of the circumcircle, is a rectangle. Hence $AB = PP'$, $AB \parallel PP'$; and similarly,

$$BC = P'P'',\ BC \parallel P'P'',\ CA = P'P'' \text{ and } CA \parallel PP''.$$

$\therefore$ $\triangle$s $PP'P''$ and ABC are congruent and have their sides respectively parallel.

$\therefore$ $\angle P'PP'' = \angle ABC$ and PR bisects $\angle P'PP''$.

Similarly, $P'R'$ and $P''R''$ are bisectors of the remaining angles of $\triangle PP'P''$. Hence, PR, $P'R'$, $P''R''$ are concurrent at J.

(2) As AB, PP'; $BC, P'P''$; $CA, P''P$ are symmetrically situated with respect to O, then I and J are also so situated; hence, $IO = JO$, J being the incenter of $\triangle PP'P''$.

(3) Let the circumcircle cut II' at L. Then $\angle I'IC = \angle IAC + \angle ACI = \angle BAL + \angle ICB = \angle BCL + \angle ICB = \angle ICL$. Hence, since $\angle ICI'$ is a right angle, L is the midpoint of II'; and O is the midpoint of IJ. Hence, JI' is parallel to OL. But as $\angle BAL = \angle LAC$, arc BL = arc LC. Hence,

$$OL \perp BC \text{ and } \therefore\ JI' \perp BC.$$

Similarly, $JI'' \perp CA$ and $JI''' \perp AB$.

(4) Arc $P''B =$ arc $P'C$. Hence, $\angle P''PB = \angle P'PC$; hence $\angle BPR = \angle RPC$. Hence, $BR : RC = BP : PC$. Similarly, $CR' : R'A = CP' : P'A$, and $AR'' : R''B = AP'' : P''B$. Hence,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = \frac{BP \cdot CP' \cdot AP''}{PC \cdot P'A \cdot P''B}.$$

But $BP = AP'$, being opposite sides of a rectangle. Similarly, $CP' = BP''$ and $AP'' = CP$. Hence,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = 1,$$

numerically. Hence, AR, BR', CR'' are concurrent.

Also solved by A. M. HARDING.

CALCULUS.

373. Proposed by C. N. SCHMALL, New York City.

In the *Encyclopaedia Britannica* article on "Capillary Action" (Vol. 5, p. 268, 11th ed.) it is shown that $1/R_1 + 1/R_2 = p/T$, in the case of a soap bubble, where R_1, R_2 are the *principal radii of curvature* at any point of the bubble; p, the difference of air-pressure; T, the energy per unit area of the film. Employing the principle that the soap bubble tends to assume a form such that the area of its surface is a *minimum* for a *given volume* of air, show by the calculus of variations that $1/R_1 + 1/R_2 = k$, a constant.

SOLUTION BY THE PROPOSER.

We have here to determine the solid which, with a given volume (capacity), contains the least surface. Hence, we have to make the surface integral

$$U = \int\int \sqrt{1 + p^2 + q^2}\, dxdy \tag{1}$$

a minimum, subject to the condition that the volume integral

$$I = \int\int z\, dxdy \tag{2}$$

is constant.

It should be remembered that in this discussion $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, and $t = \partial^2 z/\partial y^2$.

Let k be a constant. Then it is evident that, when the surface (1) is a minimum, the binomial

$$\begin{aligned}\int\int \sqrt{1+p^2+q^2}\, dxdy + \int\int kz\, dxdy \equiv \int\int (\sqrt{1+p^2+q^2} + kz) dxdy \\ \equiv \int\int V\, dxdy\end{aligned} \tag{3}$$

will also be a minimum.

Now, taking x and y as the independent variables, the condition for a mini-